Math 3235 Probability Theory

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X_1 X_2 independent

Geometric with par p_1 p_2

$$P_{X_1 X_2}(x_1, x_2) = (1-p_1)^{x_1-1}(1-p_2)^{x_2-1} p_1 p_2$$

$$Y = \min(X_1, X_2)$$

p.m.f. of Y.

$$\mathbb{P}(Y \geq y) = \mathbb{P}(\min(X_1, X_2) \geq y) =$$

$$= \mathbb{P}(X_1 \geq y \;\&\; X_2 \geq y) =$$

$$\mathbb{P}(X_1 \geq y)\,\mathbb{P}(X_2 \geq y)$$

X_i is geometric

$$\mathbb{P}(X_1 \geq y) = \sum_{z \geq y} P_Y(z) =$$

$$= \sum_{z \geq y} (1-p_1)^{z-1} p_1$$

$$= p_1 (1-p_1)^{y-1} \sum_{z=0}^{\infty} (1-p_1)^{z} =$$

$$= \underline{\underline{(1-p_1)^{y-1}}}$$

$$\mathbb{P}(Y \geq y) = (1-p_1)^{y-1} (1-p_2)^{y-1}$$

$$= [\underbrace{(1-p_1)(1-p_2)}_{\bar{q}}]^{y-1}$$

$$\mathbb{P}(Y = y) = \mathbb{P}(Y \geq y) - \mathbb{P}(Y \geq y+1)$$

$$= \bar{q}^{\,y-1} - \bar{q}^{\,y} =$$

$$= \bar{q}^{\,y-1} (1-\bar{q})$$

$$\boxed{\bar{p} = (1-\bar{q})}$$

$$= (1-\bar{p})^{y-1} \bar{p}$$

If X_1 X_2 are geometric with

par p_1 p_2

$\min(X_1, X_2)$ geometric with

par

$$\bar{p} = 1 - (1-p_1)(1-p_2)$$

$Y = \max(X_1, X_2)$ if $X_1 \sim X_2$

$$\mathbb{P}(Y \le y) = \mathbb{P}(\max(X_1, X_2) \le y) =$$

$$= \mathbb{P}(X_1 \le y)\,\mathbb{P}(X_2 \le y)$$

$$= \mathbb{P}(X_i \le y)^2$$

$$\mathbb{P}(Y = y) = \mathbb{P}(Y \le y) - \mathbb{P}(Y \le y-1) =$$

$$= \mathbb{P}(X_1 \le y)^2 - \mathbb{P}(X_1 \le y-1)^2$$

Suppose again X_1 X_2 geometric

with the same p

$X_1 \perp\!\!\!\perp X_2$

p.m.f. of $Y = X_1 + X_2$

$$\mathbb{P}(Y=y) = \sum_{\substack{x_1, x_2 \\ x_1+x_2=y}} \mathbb{P}(X_1 = x_1 \;\&\; X_2 = x_2) =$$

$$= \sum_{x_1} \mathbb{P}(X_1 = x_1)\,\mathbb{P}(X_2 = y - x_1)$$

$$p_Y(y) = \sum_x p_{X_1}(x)\, p_{X_2}(y-x)$$

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p_Y is the convolution of p_{X_1} and p_{X_2}

$$p_Y(y) = \sum_{x=1}^{y} (1-p)^{x-1} p \, (1-p)^{y-x-1} p$$

$$= p^2 (1-p)^{y-2} \sum_{x=1}^{y} 1$$

$$= y\, p^2 (1-p)^{y-2}$$

Loss of memory.

Y is geometric

$$\mathbb{P}(Y > t+s \mid Y > s) = \mathbb{P}(Y > t)$$

$$\mathbb{P}(Y \geq t) = (1-p)^t$$

$$\mathbb{P}(Y > t+s \mid Y > s) = \frac{\mathbb{P}(Y > t+s \,\&\, Y > s)}{\mathbb{P}(Y > s)}$$

$$= \frac{(1-p)^{t+s}}{(1-p)^s} = (1-p)^t$$

Poisson distribution.

cars that stop in 1 hour

is N poisson par μ

$$\mathbb{P}(N = n) = e^{-\mu} \frac{\mu^n}{n!}$$

Any car has a prob

0.1 of needing service.

M The number of car that need service in 1h.

$$\mathbb{P}(M=m) = \sum_{n \geq m} \mathbb{P}(n \text{ cars arrive})$$
$$\mathbb{P}(m \text{ need serv} \mid n \text{ arr})$$

$$\sum_{n \geq m} e^{-\mu} \frac{\mu^n}{n!} \binom{n}{m} 0.1^m \, 0.9^{n-m}$$

$$e^{-\mu} 0.1^m \sum_{n \geq m} \frac{1}{\cancel{n!}} \frac{\cancel{n!}\, \mu^n}{m!\,(n-m)!} 0.9^{n-m}$$

$$= e^{-\mu} \frac{0.1^m \mu^m}{m!} \sum_{n \geq m} \frac{\mu^{n-m} \, 0.9^{n-m}}{(n-m)!} =$$

$$= e^{-\mu} \frac{(0.1\mu)^m}{m!} \sum_{n=0} \frac{(0.9\mu)^n}{n!} =$$

$$e^{-0.1\mu} \frac{(0.1\mu)^m}{m!}$$

M is poisson par 0.1μ

P number of cars that do not need service

P is poisson par 0.9μ

$P \perp\!\!\!\perp M$

N is the number of arrivals I get if in every dt I have a prob μdt of 1 arrival and $(1-\mu)dt$ of 0 arrivals.

The prob of getting a car that need service in an interval dt is

$0.1\mu dt$

so That

M is Poisson par 0.1μ.

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if N_1 and N_2 are Poisson par μ_1 and μ_2 $\Rightarrow$

$N = N_1 + N_2$ is Poisson par $\mu_1 + \mu_2$

$$\mathbb{P}(N=n) = \sum_{m=0}^{n} \mathbb{P}(N_1 = m)\,\mathbb{P}(N_2 = n-m)$$

$$\sum_{m=0}^{n} \frac{\mu_1^m e^{-\mu_1}}{m!} \frac{\mu_2^{n-m} e^{-\mu_2}}{(n-m)!} =$$

$$= \frac{e^{-(\mu_1+\mu_2)}}{n!} \sum_{m=0}^{n} \frac{n!\,\mu_1^m \mu_2^{n-m}}{m!\,(n-m)!} =$$

$$= \frac{e^{-(\mu_1+\mu_2)}}{n!} \sum_{m=0}^{n} \binom{n}{m} \mu_1^m \mu_2^{n-m} =$$

$$= \frac{e^{-(\mu_1+\mu_2)}}{n!} (\mu_1+\mu_2)^n$$